Suggested Solutions to:
Regular Exam, Spring 2020
Industrial Organization
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Question 1: Cooperation and advertising

(a) Solve for the Nash equilibrium of the advertising game described above.

At a Nash equilibrium, each firm i maximizes its profits w.r.t. the own advertising level x_i, given the equilibrium value of the other firm's advertising level. The optimal value of x_1, given some x_2, must satisfy firm 1's first-order condition:

$$\frac{\partial \pi_1(x_1, x_2)}{\partial x_1} = 1 - \frac{x_1}{a_1} = 0 \Leftrightarrow x_1 = a_1. \tag{1}$$

That is, firm 1 has a dominant strategy: its profit-maximizing advertising level does not depend on firm 2's advertising level. Moreover, the left-hand side of the first-order condition in (1) tells us that the firm never has an incentive to set $x_1 = 0$ (because the marginal benefit of advertising is constant and equal to one, whereas the marginal cost of advertising at zero equals zero); it is therefore safe for us to ignore the non-negativity constraint on x_i.[1]

By symmetry, also firm 2 has a dominant strategy, and this is given by $x_2 = a_2$. We can conclude that the only Nash equilibrium of the game is given by:

$$(x_1^n, x_2^n) = (a_1, a_2). \tag{2}$$

(b) For what values of a_1 and a_2 is the outcome of the Nash equilibrium that you found in part (a) Pareto efficient? Prove your answer formally.

It is useful to first think about the intuition. Doing that will help us understand what we can hope to be able to prove—and how to do it. From the profit functions it is clear that advertising is associated with a positive externality. Each firm i benefits from the rival firm's advertising, but the rival will not take that benefit into account when choosing its advertising level. We should therefore expect that the advertising levels at the Nash equilibrium are too low from a social point of view.[2] Moreover, this should hold for all values of a_1 and a_2 (because the externality matters for all values of a_1 and a_2). In particular, because of the positive externality, we should expect that if both firms advertised a little bit more, they would both be better off.

[1] It is also easy to see that the second-order condition, $\frac{\partial^2 \pi_1(x_1,x_2)}{\partial x_1^2} = -\frac{1}{a_1} < 0$, is satisfied.

[2] The expression "social point of view" here refers to the two firms only, not the consumers—cf. the definition of Pareto efficiency in the question.

We can formalize the idea in the last sentence above by considering the pair of advertising levels

$$(x_1', x_2') = (a_1 + \varepsilon, a_2 + \varepsilon),$$

where ε is some possibly very small but strictly positive number. If our intuition is right, then $\pi_i(x_1', x_2') > \pi_i(x_1^n, x_2^n)$ for some $\varepsilon > 0$ and for both $i = 1$ and $i = 2$. If we can show that those two conditions indeed hold (for all values of a_1 and a_2), then we have proven that the Nash equilibrium is *not* Pareto efficient for *any* values of a_1 and a_2.

We can write

$$\begin{aligned} \pi_i(x_1', x_2') &= \pi_i(a_1 + \varepsilon, a_2 + \varepsilon) \\ &= a_1 + \varepsilon + a_2 + \varepsilon - \frac{(a_i + \varepsilon)^2}{2a_i}, \end{aligned} \tag{3}$$

where the second equality uses the profit function specified in the question (eq. (1)). Note that (i) for $\varepsilon = 0$, we obviously have $\pi_i(x_1', x_2') = \pi_i(x_1^n, x_2^n)$. Also note (ii) that by differentiating (3) w.r.t. ε, we obtain

$$\frac{\partial \pi_i(a_1 + \varepsilon, a_2 + \varepsilon)}{\partial \varepsilon} = 2 - \frac{a_i + \varepsilon}{a_i} = 1 - \frac{\varepsilon}{a_i}, \tag{4}$$

which is strictly positive for all $\varepsilon \in [0, a_i)$. Observations (i) and (ii) imply that there indeed exists some some $\varepsilon > 0$ such that both $\pi_1(x_1', x_2') > \pi_i(x_1^n, x_2^n)$ and $\pi_2(x_1', x_2') > \pi_i(x_1^n, x_2^n)$.

We can conclude, and we have proven, that the Nash equilibrium is not Pareto efficient for *any* values of a_1 and a_2.

An alternative approach that one might be tempted to try would be to identify the advertising levels x_1 and x_2 that maximize the industry profits (i.e., the sum of the two firms' profits). It is straightforward to see that these advertising levels are given by $(x_1, x_2) = (2a_1, 2a_2)$. We clearly have $(x_1^n, x_2^n) \neq (2a_1, 2a_2)$. However, this does not prove that (x_1^n, x_2^n) fails to be Pareto efficient—it just shows that (x_1^n, x_2^n) differs from *one* particular Pareto efficient pair of adverting levels. We should expect that there are many other Pareto efficient pair of adverting levels, and somehow we need to show that (x_1^n, x_2^n) differs also from all of those. Another way of seeing that the approach does not work is to note that, for large enough values of a_i, $(2a_1, 2a_2)$ does *not* Pareto dominate (x_1^n, x_2^n)—cf. (4) above.

(c) Investigate under what condition the two firms' following the above trigger strategy constitutes a subgame perfect Nash equilibrium of the infinitely repeated game. The condition should be stated as $\delta \geq K$, where K is a particular number (which must be specified).

We need to check two kinds of possible deviations from the specified strategy:

- No firm must have an incentive to deviate unilaterally along the equilibrium path. (This is a requirement for having a Nash equilibrium.)
- No firm must have an incentive to deviate unilaterally off the equilibrium path. (This is a requirement for subgame perfection.)

The requirement in the second bullet point is, by standard arguments (see lecture slides), unproblematic.

Consider the requirement in the first bullet point. Player i's total payoff for period $\widehat{t}$ and onwards if both players play the grim trigger strategy:

$$V_i^c = \sum_{t=\widehat{t}}^{\infty} \delta^{t-\widehat{t}} \pi_i^c = \pi_i^c \sum_{t=\widehat{t}}^{\infty} \delta^{t-\widehat{t}} = \frac{\pi^{ic}}{1-\delta}.$$

Player i's payoff if deviating (from the equilibrium path):

$$V_i^d = \pi_i^d + \sum_{t=\widehat{t}+1}^{\infty} \delta^{t-\widehat{t}} \pi^n = \pi_i^d + \pi_i^n \sum_{t=\widehat{t}+1}^{\infty} \delta^{t-\widehat{t}} = \pi_i^d + \frac{\delta \pi_i^n}{1-\delta}.$$

That is, player i does not have an incentive to deviate if, and only if,

$$V_i^c \geq V_i^d \Leftrightarrow \frac{\pi_i^c}{1-\delta} \geq \pi_i^d + \frac{\delta \pi_i^n}{1-\delta} \Leftrightarrow \delta \geq \frac{\pi_i^d - \pi^c}{\pi_i^d - \pi_i^n} \stackrel{\text{def}}{=} \widehat{\delta}_i. \tag{5}$$

Compute:

- We have $x_i^d = x_i^n = a_i$. This yields

  $$\pi_1^d = \pi_1(x_1^d, x_2^c) = \pi_1(2, 6) = 2 + 6 - \frac{2^2}{2(2)} = 7$$

  and

  $$\pi_2^d = \pi_2(x_1^c, x_2^d) = \pi_2(4, 3) = 4 + 3 - \frac{3^2}{2(3)} = \frac{11}{2}$$

- We have $x_i^d = x_i^n = a_i$. This yields

  $$\pi_i^n = \pi_i(x_1^n, x_2^n) = \pi_i(a_1, a_2) = a_1 + a_2 - \frac{a_i^2}{2a_i} = a_1 + a_2 - \frac{a_i}{2}.$$

  Thus,

  $$\pi_1^n = \frac{a_1}{2} + a_2 = \frac{2}{2} + 3 = 4$$

  and

  $$\pi_2^n = a_1 + \frac{a_2}{2} = 2 + \frac{3}{2} = \frac{7}{2}.$$

- We have $x_i^d = x_i^n = a_i$. This yields

  $$\pi_i^c = \pi_i(x_1^c, x_2^c) = \pi_i(2a_1, 2a_2) = 2a_1 + 2a_2 - \frac{(2a_i)^2}{2a_i} = 2(a_1 + a_2) - 2a_i.$$

  Thus,

  $$\pi_1^c = 2a_2 = 6$$

  and

  $$\pi_2^c = 2a_1 = 4.$$

Plug the results from above into the expression for $\widehat{\delta}_i$ in (5):

$$\widehat{\delta}_1 \stackrel{\text{def}}{=} \frac{\pi_1^d - \pi_1^c}{\pi_1^d - \pi_1^n} = \frac{7-6}{7-4} = \frac{1}{3}$$

and

$$\widehat{\delta}_2 \stackrel{\text{def}}{=} \frac{\pi_2^d - \pi_2^c}{\pi_2^d - \pi_2^n} = \frac{\frac{11}{2} - 4}{\frac{11}{2} - \frac{7}{2}} = \frac{3}{4}.$$

For neither firm to have an incentive to deviate, we must have both $\delta \geq \widehat{\delta}_1$ and $\delta \geq \widehat{\delta}_2$. The most stringent condition is clearly the latter.

We can thus conclude that the specified grim trigger strategy is an SPNE if, and only if,

$$\delta \geq \frac{3}{4}.$$

Question 2: Finding more equilibria in the BBPD model with a mix of naive and sophisticated consumers

(a) By studying the first-period decisions of the sophisticated consumers, derive a relationship between $\widehat{r}$ and p_1 that must hold at an equilibrium.

We are supposed to look for an equilibrium where

$$p_2^L = \frac{\gamma\widehat{r} + (1-\gamma)\, p_1}{2} \qquad \text{and} \qquad p_2^H = \widehat{r}. \tag{6}$$

Given the period 1 price p_1 and the period 2 prices p_2^L and p_2^H, a consumer with valuation r weakly prefers buying in period 1 to not doing that but buy in period 2 if, and only if,

$$r - p_1 + \delta\left[r - p_2^H\right] \geq \ 0 + \delta\left[r - p_2^L\right] \Leftrightarrow r - p_1 \geq p_2^H - p_2^L,$$

where the second inequality was obtained by rewriting and using the assumption that $\delta = 1$. For a consumer for whom $r = \widehat{r}$, the inequality must be satisfied with equality, meaning that $\widehat{r} - p_1 = p_2^H - p_2^L$.

Using (6), we can write

$$p_2^H - p_2^L = \frac{2\widehat{r}}{2} - \frac{\gamma\widehat{r} + (1-\gamma)\, p_1}{2} = \frac{(2-\gamma)\widehat{r} - (1-\gamma)\, p_1}{2}$$

Thus,

$$\widehat{r} - p_1 = p_2^H - p_2^L \Leftrightarrow 2\left(\widehat{r} - p_1\right) = (2-\gamma)\widehat{r} - (1-\gamma)\, p_1 \Leftrightarrow \widehat{r} = \frac{1+\gamma}{\gamma} p_1. \tag{7}$$

(b) By studying the firm's profit-maximization problem in period 1, find the firm's optimal choice of p_1. Then use the information about p_1 and the information in (a) to calculate the implied values of $\widehat{r}$, p_2^L and p_2^H. Investigate if these values indeed are part of an equilibrium of the model (if any conditions on the parameters are required, state these).

The firm's profit at stage 1 can be written as

$$\begin{aligned}\pi &= \pi_1 + \beta\pi_2 = \pi_1 \\ &= [\gamma(1-\widehat{r}) + (1-\gamma)(1-p_1)]\,p_1 \\ &= p_1 - [\gamma\widehat{r} + (1-\gamma)\,p_1]\,p_1 \\ &= p_1 - \left[\gamma\frac{1+\gamma}{\gamma}p_1 + (1-\gamma)\,p_1\right]p_1 \\ &= p_1 - 2p_1^2.\end{aligned} \tag{8}$$

In (8) above, the second equality follows from the assumption that $\beta = 0$; the third equality follows from the fact that first-period demand among sophisticated consumers is $1-\widehat{r}$ and among naive consumers $1-p_1$; and the fifth equality is obtained by plugging in the expression for $\widehat{r}$ from (7). The profits in (8) are thus clearly maximized at

$$p_1 = \frac{1}{4}. \tag{9}$$

What are the implied values of $\widehat{r}$, p_2^L and p_2^H? By plugging (9) into (7), we obtain

$$\widehat{r} = \frac{1+\gamma}{4\gamma}. \tag{10}$$

By plugging (9) and (10) into (6), we can write

$$p_2^L = \frac{\gamma\frac{1+\gamma}{4\gamma} + (1-\gamma)\frac{1}{4}}{2} = \frac{1}{4} \qquad \text{and} \qquad p_2^H = \frac{1+\gamma}{4\gamma}. \tag{11}$$

Can these prices and cutoff values be part of an equilibrium of the model? The answer is no. Consider the requirement that the second-period price in the L-market indeed is given by the second line of equation (3) in the exam paper:

$$p_1 \geq \frac{\sqrt{\gamma}}{1+\sqrt{\gamma}}\widehat{r}; \tag{12}$$

Plugging (9) and (10) into (12), we obtain

$$\frac{1}{4} \geq \frac{\sqrt{\gamma}}{1+\sqrt{\gamma}}\frac{1+\gamma}{\gamma}\frac{1}{4} \Leftrightarrow (1+\sqrt{\gamma})\sqrt{\gamma} \geq 1+\gamma \Leftrightarrow \sqrt{\gamma} \geq 1, \tag{13}$$

which is impossible.

(c) Derive the expression for q_2^H stated in (4). That is, explain how we can obtain this demand function, given the consumers' preferences and other assumptions that we have made. You are encouraged to use figures, if you think they can help you explain.

The second-period demand in the H-market comes potentially from two groups of consumers:

- *Naive consumers who bought in period 1 and who therefore have valuations* $r \in [p_1, 1]$. The mass of naive consumers is $1-\gamma$. (See also Figure 1.) In the second period, a consumer in this group will buy if, and only if, her valuation (weakly) exceeds the second-period H-market price: $r \in [p_2^H, 1]$. This means, as the distribution of consumer valuations is uniform, that demand coming from naive consumers is given by

$$q_{2,naive}^H = \begin{cases} (1-\gamma)\left(1-p_2^H\right) & \text{if } p_2^H \in [p_1, 1] \\ (1-\gamma)(1-p_1) & \text{if } p_2^H \in [0, p_1]. \end{cases} \tag{14}$$

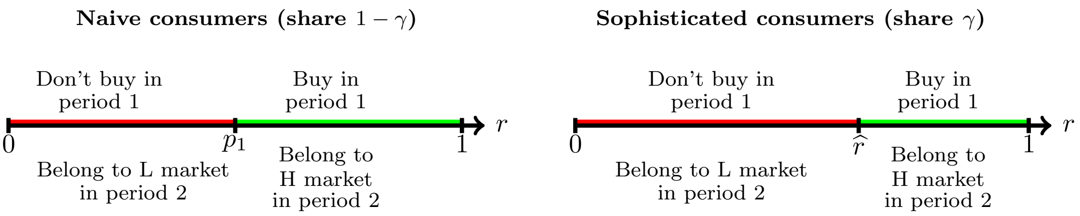

Figure 1: Period 1 behavior of naive and sophisticated consumers.

- *Sophisticated consumers who bought in period 1 and who therefore have valuations* $r \in [\widehat{r}, 1]$. The mass of sophisticated consumers is γ. (See also Figure 1.) In the second period, a consumer in this group will buy if, and only if, her valuation (weakly) exceeds the second-period H-market price: $r \in [p_2^H, 1]$. This means, as the distribution of consumer valuations is uniform, that demand coming from sophisticated consumers is given by

$$q_{2,soph}^H = \begin{cases} \gamma\left(1-p_2^H\right) & \text{if } p_2^H \in [\widehat{r}, 1] \\ \gamma\left(1-\widehat{r}\right) & \text{if } p_2^H \in [0, \widehat{r}]\,. \end{cases} \tag{15}$$

In order to get the total second-period demand in the H-market, we add up (14) and (15). When doing that, we make use of the fact that $p_1 < \widehat{r}$. This means that we have the following three intervals to consider:

- $[0, p_1]$, where the second line of (14) and the second line of (15) apply.
  - In this interval, we thus get $q_2^H = (1-\gamma)(1-p_1) + \gamma(1-\widehat{r})$.
- $[p_1, \widehat{r}]$, where the first line of (14) and the second line of (15) apply.
  - In this interval, we thus get $q_2^H = (1-\gamma)\left(1-p_2^H\right) + \gamma(1-\widehat{r})$.
- $[\widehat{r}, 1]$, where the first line of (14) and the first line of (15) apply.
  - In this interval, we thus get $q_2^H = (1-\gamma)\left(1-p_2^H\right) + \gamma\left(1-p_2^H\right) = 1-p_2^H$.

Together, the three terms yield the demand function in eq. (4) in the exam paper.